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More supersymmetric Wilson loops

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Abstract

We present a large new family of Wilson loop operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. For an arbitrary curve on the three dimensional sphere one can add certain scalar couplings to the Wilson loop so it preserves at least two supercharges. Some previously known loops, notably the 1/2 BPS circle, belong to this class, but we point out many more special cases which were not known before and could provide further tests of the *AdS/CFT* correspondence.

Supersymmetry is an extremely powerful tool in theoretical physics. In addition to the general hope and expectation that supersymmetry will be discovered at high energies, it affords the theorist extra freedom and control. Field theories with supersymmetry show similar phenomena to non-supersymmetric theories but are often easier to work with. Particularly, supersymmetric theories have special operators which are invariant under some of the supersymmetry generators and therefore belong to shorter multiplets of the algebra and may be protected from quantum corrections.

In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the local operators preserving some of the supersymmetry generators are well studied. In the dual string theory on $AdS_5 \times S^5$ they are described by the Kaluza-Klein modes arising in the reduction of the supergravity fields on S^5 , or by giant gravitons. The supersymmetric non-local operators, such as Wilson loops, are not nearly as well understood. So far the only supersymmetric Wilson loops with a non-trivial expectation value that were described had a circular geometry.

In this letter we will present a new class of Wilson loop operators that preserve between 2 and 16 supercharges. Since they are non-local they might capture interesting properties of this gauge theory. In the dual string theory they will be described by string surfaces (or D-branes) and due to their supersymmetry they may be under better computational control than non-supersymmetric Wilson loops. Indeed in a few cases their expectation values as calculated in the field theory and string theory provide an amazing test for the AdS/CFT correspondence.

We present here the main idea and several special examples of interesting loops that belong to this class. We will study those loops in greater detail in future publications [1].

In addition to the required coupling of the gauge field A_μ to the tangent vector $\dot{x}^\mu(s)$, it is natural to couple the Wilson loop in $\mathcal{N} = 4$ YM to the scalars Φ^I (with $I = 1, \dots, 6$) by the functions $\Theta^I(s)$ so it takes the form [2, 3]

$$W = \frac{1}{N} \text{Tr } \mathcal{P} \exp \oint ds (iA_\mu \dot{x}^\mu(s) + |\dot{x}| \Theta^I(s) \Phi^I) . \quad (1)$$

Requiring that such a loop be supersymmetric leads to an equation at every point along the path $x^\mu(s)$. Only if all those conditions commute, will the loop be globally supersymmetric.

One simple way to satisfy this is if at every point one finds the same equation. This happens in the case of the straight line, where \dot{x}^μ is a constant vector and one takes also Θ^I to be a constant. This idea was generalized in a very ingenious way by Zarembo [4], who assigned to every tangent vector in \mathbb{R}^4 a unit vector in \mathbb{R}^6 (through a 6×4 matrix M^I_μ) and took $|\dot{x}| \Theta^I = M^I_\mu \dot{x}^\mu$. That construction guarantees that if a curve is contained within a one-dimensional linear subspace of \mathbb{R}^4 it is 1/2 BPS. Inside

a 2-plane it will be 1/4 BPS, inside \mathbb{R}^3 it's 1/8 BPS and a generic curve is 1/16 BPS.

An amazing fact about those loops is that their expectation value seems to be trivial [5–7]. But this is also a crucial shortcoming of this construction, one of the most interesting Wilson loop observables is the circle with a coupling to a single scalar, whose expectation value is a non-trivial function of both the rank of the gauge group N and of the 't Hooft coupling $\lambda = g_{YM}^2 N$ [8, 9]. This Wilson loop preserves 1/2 of the supersymmetries, but is not given by the above construction. Recently some 1/4 BPS loops were described that also do not have trivial expectation values, rather the values of all those loops seem to be described by a 0-dimensional matrix model [10].

The supersymmetries preserved by those loops with non-unit expectation values always include the superconformal generators (written usually as S and \bar{S}) in addition to the usual Poincaré supercharges (Q and \bar{Q}). The loops with trivial expectation values are annihilated by combinations of Q and \bar{Q} alone, so our construction below gives more loops annihilated by combinations including also S and \bar{S} .

The family of Wilson loops that we present here follows an arbitrary curve on S^3 , which may be the unit sphere in flat \mathbb{R}^4 in the Euclidean gauge theory, or a spatial slice for the Lorentzian theory on $S^3 \times \mathbb{R}$. The basic ingredient in our construction are the invariant one-forms on the group manifold $SU(2) = S^3$ (we follow the conventions of [11]). In terms of flat coordinates x^μ satisfying $x^2 = 1$ they read

$$\begin{aligned}\sigma_1^{R,L} &= 2 [\pm(x^2 dx^3 - x^3 dx^2) + (x^4 dx^1 - x^1 dx^4)], \\ \sigma_2^{R,L} &= 2 [\pm(x^3 dx^1 - x^1 dx^3) + (x^4 dx^2 - x^2 dx^4)], \\ \sigma_3^{R,L} &= 2 [\pm(x^1 dx^2 - x^2 dx^1) + (x^4 dx^3 - x^3 dx^4)],\end{aligned}\tag{2}$$

where σ_i^R are the right (or left-invariant) one-forms and σ_i^L are the left (or right-invariant) one-forms. These are respectively dual to left (right) invariant vector fields ξ_i^R (ξ_i^L) generating right (left) group actions. We can now use either σ_i^R or σ_i^L to define a natural coupling to three of the scalars. We will choose to use the right one-forms. Our ansatz for the supersymmetric Wilson loop on S^3 is then

$$W = \frac{1}{N} \text{Tr } \mathcal{P} \exp \oint \left(iA + \frac{1}{2} \sigma_i^R M^i{}_I \Phi^I \right),\tag{3}$$

where for convenience we wrote the integral in form notation. The 3×6 matrix $M^i{}_I$ specifies which three scalars the loop will couple to and satisfies that MM^\top is the 3×3 unit matrix. When we need an explicit choice of M we take $M^1{}_1 = M^2{}_2 = M^3{}_3 = 1$ and all other entries zero.

The supersymmetry variation of the Wilson loop will be proportional to

$$\delta W \simeq \left(i dx^\mu \gamma_\mu + \frac{1}{2} \sigma_i^R M^i{}_I \rho^I \gamma^5 \right) \epsilon(x),\tag{4}$$

where the pullback of the one-forms along the curve is assumed. γ_μ and ρ^I are respectively the gamma matrices of $SO(4)$ and $SO(6)$, the Poincaré and R-symmetry groups and they commute with each-other. $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$ is the four dimensional chirality matrix and $\epsilon(x)$ is a conformal Killing spinor given by two arbitrary constant spinors (which are also spinors of the R-symmetry group) as

$$\epsilon = \epsilon_0 + x^\mu \gamma_\mu \epsilon_1. \quad (5)$$

Note now that the action of the γ 's on the chiral components of a spinor $\epsilon^\pm = \frac{1}{2}(1 \pm \gamma^5)\epsilon$ can be expressed in terms of the identity and Pauli matrices $\pm i\tau_i$, allowing one to write

$$i dx^\mu x^\nu \gamma_{\mu\nu} \epsilon^\mp = \pm \frac{1}{2} \tau^i \sigma_i^{R,L} \epsilon^\mp. \quad (6)$$

Therefore if we restrict to the anti-chiral components of ϵ , equation (4) can be written as

$$\delta W \simeq \frac{1}{2} \sigma_i^R (\tau^i \epsilon_1^- - M^i{}_I \rho^I \epsilon_0^- - x^\eta \gamma_\eta (\tau^i \epsilon_0^- - M^i{}_I \rho^I \epsilon_1^-)). \quad (7)$$

This equation is solved if

$$\tau^i \epsilon_1^- = M^i{}_I \rho^I \epsilon_0^-. \quad (8)$$

On the other hand, the chiral part of ϵ will introduce σ_i^L into (4), so unless there are linear relations between the six $\sigma_i^{R,L}$ and/or $x^\eta \gamma_\eta$, one finds that $\epsilon^+ = 0$. While this is true for a generic curve on S^3 , some of the special examples we detail below are degenerate and posses extra supersymmetries.

To solve the set of equations (8), let us choose the matrix M that identifies i with I . Then we can eliminate ϵ_0^- to get

$$i\tau_i \epsilon_1^- = -\frac{1}{2} \epsilon_{ijk} \rho^{jk} \epsilon_1^-, \quad i = 1, 2, 3. \quad (9)$$

This is a consistent set of three constraints, out of which only two are independent. Since ϵ_1^- has eight real components and the other spinor, ϵ_0^- , is determined from it, we conclude that for a generic curve on S^3 the Wilson loop preserves 1/16 of the original supersymmetries.

To find explicitly the combinations of \bar{Q} and \bar{S} which leave the Wilson loop invariant, notice that singling out three of the scalars breaks the R-symmetry group $SU(4)$ down to $SU(2)_A \times SU(2)_B$, where $SU(2)_A$ acts on Φ^1, Φ^2, Φ^3 while $SU(2)_B$ rotates Φ^4, Φ^5, Φ^6 . The operators appearing in (9) are just the generators of $SU(2)_R$ and $SU(2)_A$, and the above equations simply state that ϵ_1^- is a singlet of the diagonal sum of $SU(2)_R$ and $SU(2)_A$. If we split the $SU(4)$ index A into \dot{a} for $SU(2)_A$ and a for $SU(2)_B$, then the singlet can be written as

$$\epsilon_1^a = \varepsilon^{\dot{a}\dot{a}} \epsilon_{1,\dot{a}\dot{a}}^a. \quad (10)$$

Using any of the equations in (8) we can determine $\epsilon_0^- = \tau_3 \rho^3 \epsilon_1^- = -\epsilon_1^-$.

This suggests that those operators can be defined in a topologically twisted version of $\mathcal{N} = 4$ SYM. This twisting consists of replacing the $SU(2)_R$ factor in the Lorentz group with the sum of $SU(2)_R$ and $SU(2)_A$ [12,13]¹. After the twisting the supercharges decompose under the three remaining $SU(2)$ factors (the middle one is the twisted group) as

$$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{2}) \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}). \quad (11)$$

From the above it is clear that the two supercharges that annihilate our Wilson loops are in the $(\mathbf{1}, \mathbf{1}, \mathbf{2})$, so they become scalars after the twisting. As usual, one would then like to regard them as BRST charges, and the Wilson loops will be observables in their cohomology.

Note though that in our case the charges consist of a combination of \bar{Q} and \bar{S}

$$\bar{Q}^a = \varepsilon^{\dot{\alpha}\dot{a}} (\bar{Q}_{\dot{\alpha}\dot{a}}^a - \bar{S}_{\dot{\alpha}\dot{a}}^a), \quad (12)$$

so their anti-commutators do not vanish, but close on the $SU(2)_B$ generators, which complicates somewhat their identification with BRST charges.

To illustrate the richness of this construction we present now seven different subclasses of operators which preserve more supersymmetries. Each example has some special features that we point out here and will elaborate on in [1].

1. **Large S^2 :** If we restrict the loop to lie on an S^2 defined by, say, $x^4 = 0$, then the left and right forms are no longer independent, rather

$$\sigma_i^L = -\sigma_i^R = -2\epsilon_{ijk} x^j dx^k. \quad (13)$$

Then (4) has more solutions. In addition to the anti-chiral supersymmetries written above, such a Wilson loop also preserves two chiral supersymmetries. The generic Wilson loop on S^2 will therefore preserve 1/8 of the supersymmetries.

There is an interesting property of the loops on S^2 involving the replacement of the gauge and scalar couplings. Consider an arbitrary smooth curve on S^2 which is nowhere a geodesic parameterised by $\vec{x}(s)$, and let us take $|\dot{x}| = 1$. The scalar couplings will be given by the standard cross product in three dimensions as $\vec{\Theta}(s) = \dot{\vec{x}} \times \vec{x}$. Those are also unit vectors in \mathbb{R}^3 , so we can consider also a loop whose shape is given by $\vec{\Theta}$. A simple calculation shows that the scalar couplings for the new loop will be proportional to \vec{x} .

¹It is case ii) in [13]. The loops of [4] are related to case i), see [7].

This suggests the existence of a duality between the scalar and vector couplings and it is tempting to speculate that it will extend to a duality between the embedding of the string in the dual description into the AdS_5 and S^5 parts of the geometry.

2. **Large circle:** By this construction a maximal circle will couple only to a single scalar, for example a circle in the $(1, 2)$ plane will couple only to Φ^3 . Studying the supersymmetry variation leads to the single constraint

$$\rho^3 \gamma^5 \epsilon_0 = i \gamma_{12} \epsilon_1, \quad (14)$$

so the loop preserves 16 (8 chiral and 8 anti-chiral) supercharges. This is the most studied 1/2 BPS circular Wilson loop, whose perturbative expansion seems to be captured by a Gaussian matrix model [8, 9].

3. **Latitude:** Consider a non-maximal circle (a latitude) parametrized by

$$x^\mu = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0, 0). \quad (15)$$

This is essentially the same Wilson loop operator considered in [14, 10], except that by a conformal transformation we moved the circle from the equator to a parallel². Here the scalar couplings also follow a latitude on the “dual” S^2 , but at $\pi/2 - \theta_0$ instead of θ_0 .

If $\cos \theta_0 \neq 0$, the loop preserves 1/4 of the supersymmetries. This loop also seems to be given by a Gaussian matrix model with the only modification that the coupling g^2 is replaced by $g^2 \sin^2 \theta_0$.

4. **Two longitudes:** Inside a large S^2 consider a loop made of two arcs of length π connected at an arbitrary angle δ , i.e. two longitudes on the sphere. We can parametrize the loop in the following way

$$\begin{aligned} x^\mu &= (\sin t, 0, \cos t, 0), & 0 \leq t \leq \pi, \\ x^\mu &= (-\cos \delta \sin t, -\sin \delta \sin t, \cos t, 0), & \pi \leq t \leq 2\pi. \end{aligned} \quad (16)$$

The corresponding Wilson loop operator will couple to Φ^2 along the first arc and to $-\Phi^2 \cos \delta + \Phi^1 \sin \delta$ along the second one. Each arc, being (half) a maximal circle, produces a single constraint and is 1/2 BPS. Together the combined system is 1/4 BPS.

This example has many interesting features. By a stereographic projection it is mapped to a cusp in the plane, where along each of the rays the scalar coupling

²Also compared to [10] θ_0 is replaced here by $\pi/2 - \theta_0$.

is constant. This is an operator of the class constructed in [4] and has trivial expectation value. The observable on S^2 is not trivial, rather [1]

$$W \simeq \begin{cases} 1 + \frac{g^2 N}{8\pi^2} \delta(2\pi - \delta), & g^2 N \ll 1, \\ \exp \sqrt{\frac{g^2 N}{\pi^2}} \delta(2\pi - \delta), & g^2 N \gg 1, \end{cases} \quad (17)$$

Note that as in the latitude case, the only modification from the circle at $\delta = \pi$ is the rescaling of the coupling both at weak and at strong coupling by the same factor. But here the perturbative calculation does not seem as simple as before.

5. **Hopf fibers:** The large circle in the $(1, 2)$ plane, mentioned above, coupled only to the scalar Φ^3 . There is actually a 2-parameter family of circles that will couple to this same scalar. They are the integral curves of the vector field ξ_3^R dual to σ_3^R . Each of those circles is a fiber in the Hopf-fibration of S^3 , where the base is an S^2 .

While a single circle preserve 16 supercharges, the combined system of two or more fibers will break all the chiral supercharges and will preserve the 8 anti-chiral ones. So this system too is 1/4 BPS. If one considers this system in perturbation theory, the effective propagator, including both the gauge field and scalar exchange, between two arbitrary points along two of those circles will be a constant, independent of the distance between the circles. This suggests that this system is also described by the Gaussian matrix model, and that there is a “no-force” condition between two circles when they are moved parallel to the Hopf-fibration.

6. **Anti-chiral 1/8 BPS loops:** In a similar way to the last example, where adding more circles broke the chiral supersymmetry it is possible to break the chiral supersymmetries preserved by the 1/4 BPS longitudes example. There is a family of arbitrary curves (except for one constraint) on the base of the Hopf-fibration. The idea is straight-forward, but writing down the details is a bit complicated, so we will leave it to [1].
7. **Infinitesimal loops:** Finally, if a loop is very small, concentrated entirely near a single point, say $x^4 = 1$, one will not see the curvature of the sphere anymore. The left and right forms will then become exact differentials

$$\sigma_i^{R,L} \sim 2dx_i, \quad i = 1, 2, 3. \quad (18)$$

So the Wilson loops (3) reduces to the ones constructed by Zarembo in [4] (note that this construction does not allow for an arbitrary curve in \mathbb{R}^4 , but only in \mathbb{R}^3). This may explain why in this case the expectation value of the loops vanishes. The planar loops come from infinitesimal ones on S^3 , so it is quite natural that their VEVs vanish.

We have presented a new class of supersymmetric Wilson loops in $\mathcal{N} = 4$ supersymmetric YM. All the examples above, except for the large circle, the latitudes and the infinitesimal curves were not known before. Those operators will generally have non-vanishing expectation values, providing a wide arena for possible calculations both on the gauge theory side and in string theory and may lead to further tests of the *AdS/CFT* correspondence, as well as being perhaps related to topological gauge theory. In addition such loops may have applications in theories with less supersymmetry.

More details on this construction as well as explicit gauge theory and string theory calculations will be provided in an upcoming publication [1].

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